Majority-vote model on a random lattice

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The stationary critical properties of the isotropic majority vote model on random lattices with quenched connectivity disorder are calculated by using Monte Carlo simulations and finite size analysis. The critical exponents γ and β are found to be different from those of the Ising and majority vote on the square lattice model and the critical noise parameter is found to be $q_c = 0.117 \pm 0.005$.

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I. INTRODUCTION

Random lattices play an important role in the description of idealized statistical geometry in a great variety of fields [1–4]. Besides its potential to describe the topology in several condensed matter problems, the comparison of the universality class of systems between random and regular lattices is also the subject of intense research [5–9]. The randomness considered in these studies were either by random variation of the coupling strengths, random deletion of bonds or sites, or by spatially correlated lattices. In particular, according to the Harris criterion [7] valid for the random-bond paradigm, random disorder is marginally important to the two-dimensional Ising model since the specific heat critical exponent is $\alpha = 0$. However, this criterion could not be applied to lattices with a nonperiodic coordination number. For these models, Luck [10] formulated a criterion to such cases. For example, Janke *et al.* [11], using the single-cluster Monte Carlo update algorithm [12], reweighting techniques [13], and size scaling analysis [14], simulate the Ising model in a two-dimensional Voronoy-Delaunay random lattice. They calculated the critical exponents and found that this random system belongs to the same universality class as the pure-two-dimensional ferromagnetic Ising model.

The Voronoi-Delaunay network is a type of disordered lattice exhibiting a random coordination number that varies from 3 to ∞, depending on the number of sites. In addition, the distance r between nearest neighbors changes in a random way from site to site. Lima *et al.* used a two-dimensional Voronoi-Delaunay lattice to study the ferromagnetic Ising model [15] and the Potts model [16]. As the bond length between the first neighbors varies randomly from neighbor to neighbor, they considered a coupling factor decaying exponentially as

$$J_{ij} = J_0 \exp(-a r_{ij}), \tag{1}$$

where r_{ij} is the relative distance between sites i and j, J_0 is a constant, and $a \geqslant 0$ is a model parameter. In Ref. [15] they calculated the critical point exponents γ/ν, β/ν, and ν, and concluded that this random system belongs to the same universality class as the pure-two dimensional ferromagnetic Ising model. In Ref. [16] they studied the three-state Potts model and found that critical exponents γ and ν are different from the respective exponents of the three-state Potts model on a regular square lattice. However, a ratio γ/ν remains essentially the same. They also found numerical evidences that the specific heat on this random system behaves as a power law for $a = 0$ and as a logarithmic divergence for $a = 0.5$ and $a = 1.0$.

It has been argued that nonequilibrium stochastic spin systems on regular square lattice with up-down symmetry fall in the universality class of the equilibrium Ising model [17]. This conjecture was found in several models that do not obey detailed balance [18–20]. Campos *et al.* [21] investigated the majority-vote model on small-world network by rewiring the two-dimensional square lattice. These small-world networks, aside from presenting quenched disorder, also possess long-range interactions. They found that the critical exponents γ/ν and β/ν are different from the Ising model and depend on the rewiring probability. However, it was not evident that the exponents change was due to the disordered nature of the network or due to the presence of long-range interactions.

Here, we analyze a relatively simple nonequilibrium model with up-down symmetry in a random lattice with quenched *connectivity* disorder $(a = 0)$, namely the isotropic majority vote model [22]. Our main motivation is to investigate whether only the presence of quenched lattice disorder is capable of modifyng the exponents γ/ν and β/ν using the Voronoi-Delaunay random lattice. Our numerical results suggest that the critical exponents, in the stationary state, are different from those of the Ising model and the isotropic majority vote model on a square lattice. In what follows we will utilize Monte Carlo simulations and finite-size analysis.

II. MODEL AND SIMULATION

For each point in a given set of points in a plane, we determine the polygonal cell that contains the region of space nearest to that point than any other. We considered two cells neighbors when they possess an extremity in common. From this Voronoi tessellation, we can obtain the dual lattice by the following procedure.

(a) When two cells are neighbors, a link is placed between the two points located in the cells.

(b) From the links, we obtain the triangulation of space that is called the Delaunay lattice.

(c) The Delaunay lattice is dual to the Voronoi tessellation in the sense that its points correspond to cells, links to edges and triangles to the vertices of the Voronoi tessellation. We consider a two-dimensional majority vote model, on this Poissonian random lattice, defined [22,23] by a set of "voters" or spins variables $\{\sigma_i\}$ taking the values +1 or −1, situated on every site of a Delaunay lattice with N sites and periodic boundary conditions, and evolving in time by single spin-flip like dynamics with a probability w_i given by

$$w_i(\sigma) = \frac{1}{2}\left[1 - (1-2q)\sigma_i S\left(\sum_{\delta} \sigma_{i+\delta}\right)\right], \tag{2}$$

where $S(x)=\mathrm{sgn}(x)$ if $x \neq 0$, $S(x)=0$ if $x=0$, and the sum runs over all nearest neighbors of σ_i. In this lattice, the coordination number varies locally between 3 and ∞. The control parameter q plays the role of temperature in equilibrium systems and measures the probability of aligning antiparallel to the majority of neighbors.

For simplicity, the length of the system is defined here in terms of the size of a regular lattice, $L=N^{1/2}$. We perform simulations for different lattice sizes $N=2^i$, where i varies from 8 to 14. For each size, we generated 50 randomly chosen lattice realizations, where each simulation started with a random configuration of spins. From a given configuration, the next one was obtained as follows. (a) Choose a spin $\{i\}$ at random. (b) Generate a random number r uniformly distributed between zero and one. (c) Flip spin i when $r < w_i(\sigma)$. In our simulations, 5×10^4 Monte Carlo steps were required for attaining the stationary state. After that, we estimated the averages, for any lattice size, using 10^5 Monte Carlo steps.

We define the variable $m=\sum_{i=1}^{N}\sigma_i/N$. In particular, we were interested in the magnetization [22,24], susceptibility and the reduced fourth-order cumulant:

$$m(q) = [\langle |m| \rangle]_{av}, \tag{3}$$

$$\chi(q) = N[\langle m^2 \rangle - \langle |m| \rangle^2]_{av}, \tag{4}$$

$$U(q) = \left[1 - \frac{\langle m^4 \rangle}{3\langle |m| \rangle^2}\right]_{av}, \tag{5}$$

where $\langle \cdots \rangle$ stands for a thermodynamics average and $[\cdots]_{av}$ square brackets for averages over the 50 realizations. We calculated the error bars from the fluctuations among realizations. Note that these errors contain both, the average thermodynamic error for a given realization and the theoretical variance for infinitely accurate thermodynamic averages which is caused by the variation of the quenched random geometry of the 50 lattices.

These quantities are functions of the noise parameter q and obey the finite-size scaling relations

$$[\langle |m| \rangle]_{av} = L^{-\beta/\nu} f_m(x)[1 + \cdots], \tag{6}$$

$$\chi = L^{-\gamma/\nu} f_\chi(x)[1 + \cdots], \tag{7}$$

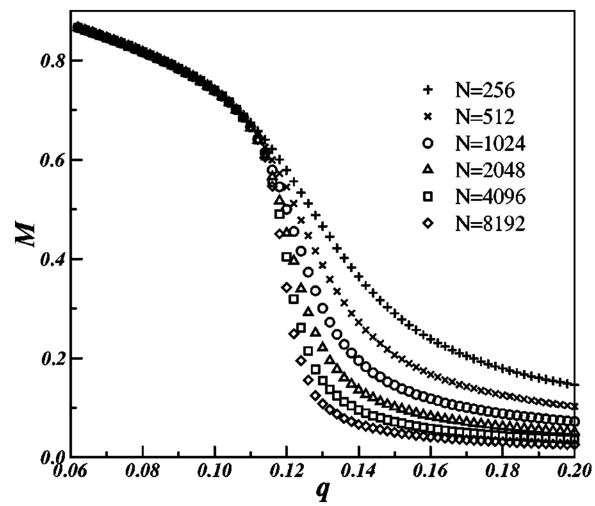

FIG. 1. Magnetization $m(q)$ as a function of the noise parameter q for several values of the system size N.

$$\frac{dU}{dq} = L^{1/\nu} f_U(x)[1 + \cdots], \tag{8}$$

where ν, β, and γ are the usual critical exponents, $f_i(x)$ are the finite size scaling functions with

$$x = (q - q_c)L^{1/\nu} \tag{9}$$

being the scaling variable, and the brackets $[1+\cdots]$ indicate corrections-to-scaling terms.

III. RESULTS

In Fig. 1 the magnetization is shown as a function of the noise parameter (q) for several values of L. As can be noticed there is a phase transition from an ordered state $(M_L > 0)$ to a disordered state $(M_L \approx 0)$. This figure displays that for $q > q_c$ the magnetization disappears when larger values of L are considered, whereas it reaches a finite value for q

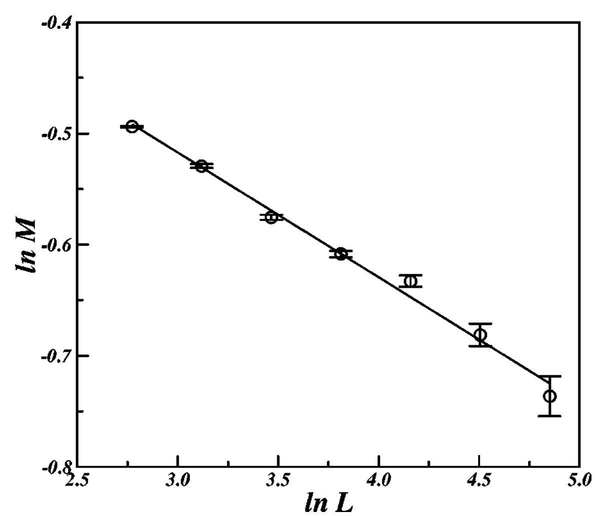

FIG. 2. Log-log plot of the magnetization at $q=q_c$ versus L. The solid line is the best fit with slope $-\beta/\nu=-0.112(4)$.

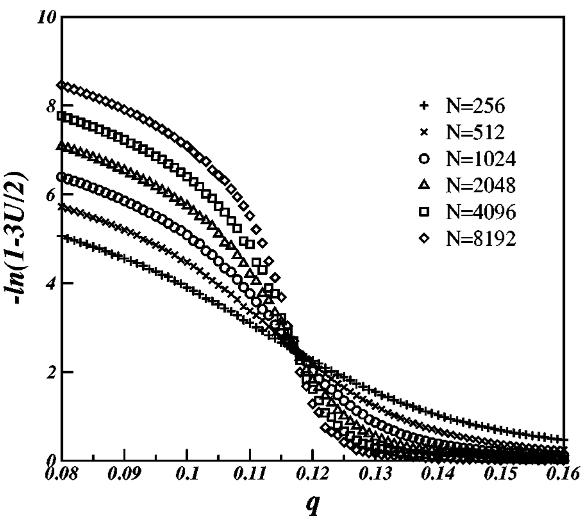

FIG. 3. Reduced fourth-order cumulant $U(q)$ as a function of q for several values of N. Within the accuracy of our data, all curves intersect at q_c=0.117(5). The value of $U(q)$ at the intersection is U^*=0.61(2).

$<q_c$. In Fig. 2 a log-log plot of the magnetization at $q=q_c$ versus L is shown. The ratio between the critical exponent $\beta/\nu=0.112\pm0.004$ is the slope of the straight line fitted to the data points. Within the numerical accuracy, we found that these exponents are distinct from the exponents characterizing the class of universality of the equilibrium Ising model [22].

To determinate the critical point the reduced fourth-order cumulant U is plotted against q. In Fig. 3, the critical point q_c is estimated when all curves, for different sizes L, intersect in the same point. We get $q_c=0.117\pm0.005$ and $U^*=0.61\pm0.02$. The value of U^* is, considering the error bar, the same as the one obtained for Ising model on a square lattice with periodic boundary conditions $U^*=0.611\pm0.001$. Another way of getting the critical point is through the relation of scale $q^{\chi}_{max}(L)=q_c+aL^{-1/\nu}$, where we get $q_c=0.117\pm0.003$ using $\nu=1.06$.

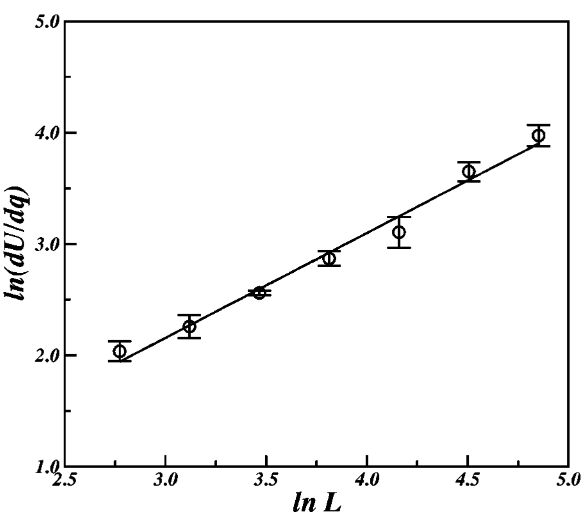

FIG. 4. Log-log plot of $dU(q)/dq$ at $q=q_c$ versus L. The solid line is the best fit with slope $1/\nu$=0.94(6).

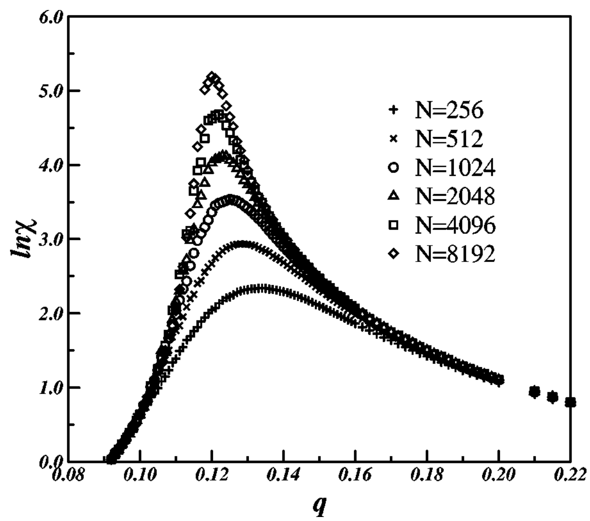

FIG. 5. Susceptibility $\chi(q)$ as function of q for several values of N.

To obtain the critical exponent ν, we calculated numerically $U'(q)=dU(q)/dq$ at the critical point for each value of L. The results are in good agreement with the scaling relation (7). Then, we plotted ln U' versus ln L, as displayed in Fig 4. The straight line represents the best fit to the data points. The slope gives $1/\nu=0.94\pm0.06$, which corresponds to $\nu=1.06\pm0.08$.

In Fig. 5, we have ln $\chi(q)$ as a function of q for several values of L. In order to study the universality of the model, the ratio γ/ν was estimated from the log-log plot of the value of the susceptibility $\chi(q_c)$ versus L. We estimated the critical exponent $\gamma/\nu=1.51\pm0.04$ from the best fit of data points, as displayed in Fig. 6. It is worth mentioning that in Figs. 1, 3, and 5 we do not include $L=128$ because of its large computational demand.

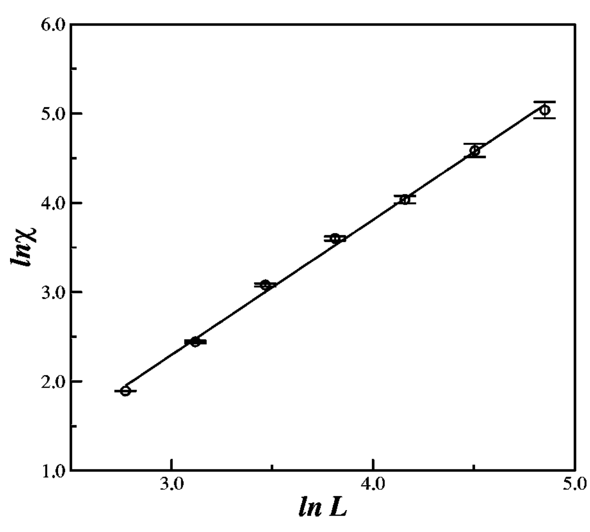

FIG. 6. Log-log plot of the susceptibility at q_c versus L. From it we estimate the critical exponent γ/ν=1.51(4) as the best fit of data points.

IV. CONCLUSION

We have studied the majority-vote model on Voronoi-Delaunay random lattices with periodic boundary conditions. These lattices possess natural quenched disorder in their connections. We verified whether only this type of disorder is significant to obtain critical exponents different of those found for this model in the regular lattice (that have the same exponents of the Ising model in two dimensions). We measure the exponents γ/ν, β/ν, and ν. The best fit of these exponents provided $\nu=1.06\pm0.08$, $\gamma/\nu=1.51\pm0.04$, and $\beta/\nu=0.112\pm0.004$ and $U^*=0.61\pm0.02$. The critical exponents β/ν and γ/ν are different from the exact values of the Ising model and majority-vote model on a regular square lattice [22]. Our results are in agreement with the results obtained by Campos *et al.* [21] that studied this same model on a small world network presenting quenched disorder and long-range interactions. They found that the critical exponents depends on the shortcuts introduced in the network. In summary we showed here that the presence of quenched connectivity disorder is enough to alter the exponents β/ν and γ/ν of the pure model and therefore that disorder is a relevant term to such non-equilibrium phase-transition. However, the critical value of the fourth-order cumulant remains the same as that of the pure model.

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